Representation of Small Conformal Algebra in κ-basis

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Abstract

In hep-th/0202087 it was argued that the operator L_0 is bad defined in κ-basis as a kernel operator. Indeed, we show that L_0 is a difference operator. We also find a representation of L_1 and L_{-1} in a class of difference operators.

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1 Introduction

The basic ingredients in the construction of the covariant string field theory are Witten's star product [1] and BRST operator. Recent progress in the diagonalization [2] of the Neumann matrices M'^{rs} defining star product allows one to identify this product with the continuous Moyal product [3, 4]. We have

$$\sum_{n=1}^{\infty} M'^{rs}_{mn} v_n^{(\kappa)} = \mu^{rs}(\kappa) v_m^{(\kappa)}$$

where $-\infty < \kappa < \infty$, the eigenvalues $\mu^{rs}(\kappa)$ are

$$\mu^{rs}(\kappa) = \frac{1}{1 + 2\cosh\frac{\pi\kappa}{2}} \Big[1 - 2\delta_{r,s} + e^{\frac{\pi\kappa}{2}} \delta_{r+1,s} + e^{-\frac{\pi\kappa}{2}} \delta_{r,s+1} \Big].$$

The eigenvectors $v_n^{(\kappa)}$ are given by their generating function

$$f^{(\kappa)}(z) = \sum_{n=1}^{\infty} \frac{v_n^{(\kappa)}}{\sqrt{n}} z^n = \frac{1}{\kappa\sqrt{\mathcal{N}(\kappa)}} (1 - e^{-\kappa \tan^{-1} z}) \tag{1.1}$$

where

$$\mathcal{N}(\kappa) = \frac{2}{\kappa} \sinh\left(\frac{\pi\kappa}{2}\right).$$

In spite of the fact that we understand the nature of the star product, it is still unclear how BRST operator acts in the basis in which star product is simple. Moreover we do not know a representation of Virasoro generators in this basis. The authors of [3] tried to construct generator L_0 us a kernel operator in the κ-basis. They show that this kernel is *not* defined as a distribution. In this paper I show that small conformal algebra $\{L_{-1},\, L_0,\, L_1\}$ is represented by a certain difference operators in the κ-basis.

Let me briefly remind a construction of the Fock space at hand [5]. In the discrete basis we have creation and annihilation operators $a_n^\dagger$ and a_n, $n = 1, 2, \ldots$. Introduce the corresponding operators in the continuous basis via

$$a_\kappa^\dagger = \sum_{n=1}^{\infty} v_n^{(\kappa)} a_n^\dagger \quad \text{and} \quad a_\kappa = \sum_{n=1}^{\infty} v_n^{(\kappa)} a_n. \tag{1.2}$$

Let $f_1(\kappa), \ldots, f_m(\kappa)$ be functions from Schwartz space, then the states

$$a^\dagger(f_1) \ldots a^\dagger(f_m)|0\rangle, \quad \text{where} \quad a^\dagger(f) = \int_{-\infty}^{\infty} d\kappa\, f(\kappa) a_\kappa^\dagger \tag{1.3}$$

form a dense subset in this Fock space. Since the operator

$$L_0 = \sum_{n=1}^{\infty} n a_n^\dagger a_n \tag{1.4}$$

contains only one creation and one annihilation operator it is completely determined by specifying its action on one particle states. Therefore we can translate the action of operator L_0 on states to its action on functions from the Schwartz space via

$$[L_0,\, a^\dagger(f)] = a^\dagger(L_0[f]). \tag{1.5}$$

In the next section I show that

$$L_0[f](\kappa) = \frac{1}{4}\left[\sqrt{\kappa(\kappa+i2^-)}\, f(\kappa+i2^-) + \sqrt{\kappa(\kappa-i2^-)}\, f(\kappa-i2^-)\right], \tag{1.6}$$

where $2^- = 2 - 0$. The operator (1.6) is defined on a class of holomorphic functions in the strip $-2 < \Im\kappa < 2$. These functions may have poles on the boundary, and the term $i2^-$ shows how one approaches these poles. I.e. one should approach the pole on the line $\kappa + 2i$ from the bottom and the pole on the line $\kappa - 2i$ from the top. If it happens that the function f is holomorphic in strip $-4 < \Im\kappa < 4$ one can apply operator (1.6) once again, etc.

Notice that the operators L_1 and L_{-1} also contain only one creation and one annihilation operator, and therefore they are completely determined by specifying their action on one particle subspace. We can write

$$[L_{\pm 1},\, a^\dagger(f)] = a^\dagger(L_{\pm 1}[f]), \tag{1.7}$$

where

$$L_1[f] = -\frac{\kappa}{2}\, f(\kappa) + \frac{i}{4}\left[\sqrt{\kappa(\kappa+i2^-)}\, f(\kappa+i2^-) - \sqrt{\kappa(\kappa-i2^-)}\, f(\kappa-i2^-)\right] \tag{1.8a}$$

$$L_{-1}[g] = -\frac{\kappa}{2}\, f(\kappa) - \frac{i}{4}\left[\sqrt{\kappa(\kappa+i2^-)}\, f(\kappa+i2^-) - \sqrt{\kappa(\kappa-i2^-)}\, f(\kappa-i2^-)\right] \tag{1.8b}$$

The paper is organized as follows. In Sections 2 and 3 we give a derivation of the equations (1.6) and (1.8) correspondingly. In Section 4 we check that the generators (1.6) and (1.8) defining in a class of difference operators have indeed correct commutation relations. The Appendix contains the technical information.

2 Operator L_0

Using the fact that

$$[L_0, a_n^\dagger] = n a_n^\dagger. \tag{2.1}$$

one can easily obtain that L_0 acts on the generating function (1.1) (it should be considered as function of κ; z is an external parameter) in the following way

$$L_0[\hat{f}^{(\kappa)}] = z\frac{d}{dz}\,\hat{f}^{(\kappa)}(z). \tag{2.2}$$

From this one can easily obtain a formal expression for the kernel of the operator L_0

$$L_0(\kappa,\kappa') \sim \sum_{n=1}^{\infty} n v_n^{(\kappa)} v_n^{(\kappa')}. \tag{2.3}$$

It was argued in [3] that this kernel is not defined as a distribution. This means that the operator L_0 is not a kernel operator in the κ-basis.

It was explained in the Introduction that operator L_0 is completely determined by its restriction onto the one-particle Fock space. To actually find it we will use the following strategy. First, find the inverse of the operator L_0 on one particle Fock subspace. In other words we are looking for an operator G_0 such that

$$G_0 L_0|\psi\rangle = L_0 G_0|\psi\rangle = |\psi\rangle$$

for any state $|\psi\rangle = a^\dagger(f)|0\rangle$ specifying by function f from Schwartz space. Using relation (1.5) one can formulate this statement in the Schwartz space as

$$G_0[L_0[f]](\kappa) = L_0[G_0[f]](\kappa) = f(\kappa) \tag{2.4}$$

for any Schwartz function $f(\kappa)$. From (2.3) one can easily obtain a formal expression for the kernel of operator G_0

$$G_0(\kappa,\kappa') = \sum_{n=1}^{\infty} \frac{1}{n} v_n^{(\kappa)} v_n^{(\kappa')}. \tag{2.5}$$

Straight forward calculations of this kernel, which I present in Appendix A, show that operator G_0 is indeed a kernel operator and

$$G_0(\kappa,\kappa') = \left[\frac{\theta(\kappa)}{\kappa}\right]^{1/2} \left[\frac{\theta(\kappa')}{\kappa'}\right]^{1/2} \frac{1}{4\cosh\left[\frac{\pi}{4}(\kappa-\kappa')\right]}. \tag{2.6}$$

Here $\theta(\kappa) = 2\tanh\frac{\pi\kappa}{4}$ is a non-commutativity parameter specifying the continuous Moyal algebra [3].

Second, we find the inverse of operator G_0 on the one-particle Fock space. In other words we are going to solve equation (2.4). The resulting operator will coincide with the operator L_0 on the one-particle Fock space. But L_0 is completely determined by its action on this subspace and therefore we will actually find operator L_0 on the whole Fock space.

So we need to solve the equation $G_0[L_0[f]] = f$:

$$\int_{-\infty}^{\infty} d\kappa' \left[\frac{\theta(\kappa)}{\kappa}\right]^{1/2} \frac{1}{4\cosh\left[\frac{\pi}{4}(\kappa-\kappa')\right]} \left[\frac{\theta(\kappa')}{\kappa'}\right]^{1/2} g(\kappa') = f(\kappa), \tag{2.7}$$

where $g = L_0[f]$. It is convenient to introduce new functions G and F via

$$G(\kappa') = \left[\frac{\theta(\kappa')}{\kappa'}\right]^{1/2} g(\kappa') \quad \text{and} \quad F(\kappa) = \left[\frac{\theta(\kappa)}{\kappa}\right]^{-1/2} f(\kappa). \tag{2.8}$$

Then equation (2.7) takes a from

$$\int_{-\infty}^{\infty} d\kappa' \, \frac{1}{4\cosh\left[\frac{\pi}{4}(\kappa-\kappa')\right]} \, G(\kappa') = F(\kappa). \tag{2.9}$$

This equation can be easily solved by using Fourier transformation method:

$$F(\kappa') = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \, e^{i\omega\kappa'} \tilde{F}(\omega)$$

Using that

$$\int_{-\infty}^{\infty} d\kappa' \, \frac{e^{i\omega\kappa'}}{4\cosh\left[\frac{\pi}{4}(\kappa-\kappa')\right]} = \frac{e^{i\omega\kappa}}{\cosh 2\omega}$$

one obtains

$$\tilde{G}(\omega) = \cosh 2\omega \, \tilde{F}(\omega).$$

Finally the solution to (2.9) is of the form

$$G(\kappa) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\kappa' \, e^{i\omega(\kappa-\kappa')} \cosh 2\omega \, F(\kappa'). \tag{2.10}$$

The integral over ω does not exist (this once again confirms that L_0 is not a kernel operator) and therefore we can not simply switch integrals over ω and over κ'. One can easily see that the integral operator in the right hand side is well defined at least on functions on κ' with Gaussian fall off at infinity. One can give a precise mathematical meaning to this operator if instead of Fourier transform one will use a Laplace transform and define $0 < \kappa < \infty$. In this case we can rigorously prove that[1]

$$G(\kappa) = \frac{1}{2}\Big[F(\kappa + 2i - i0) + F(\kappa - 2i + i0)\Big]. \tag{2.11}$$

The function F here is supposed to be holomorphic function in the strip $-2 < \Im\kappa < 2$. It is allowed to have poles on the boundary of the strip. The $\pm i0$ in the formula above tells us how one should approach this poles. Namely one has to approach the pole on the line $\kappa + 2i$ from the bottom and the one on the line $\kappa - 2i$ from the top. Substitution of (2.8) into (2.11) yields expression (1.6) for the operator L_0.

Let us now check that (2.11) is indeed a solution to equation (2.9):

$$\begin{aligned}
&\int_{-\infty}^{\infty} d\kappa' \, \frac{1}{4\cosh\left[\frac{\pi}{4}(\kappa-\kappa')\right]} \, \frac{1}{2}\Big[F(\kappa + 2i - i0) + F(\kappa - 2i + i0)\Big] \\
&= \frac{1}{8}\int_{-\infty}^{\infty} d\kappa' F(\kappa') \left[\frac{i}{\sinh\frac{\pi}{4}(\kappa-\kappa') + i\varepsilon\cosh\frac{\pi}{4}(\kappa-\kappa')} + \frac{-i}{\sinh\frac{\pi}{4}(\kappa-\kappa') - i\varepsilon\cosh\frac{\pi}{4}(\kappa-\kappa')}\right] \\
&= F(\kappa).
\end{aligned}$$

[1] Of course, expression (2.11) can be obtained by a formal integration of (2.10).

Here in the middle line we have made a shift of integration variable, which is allowed due to the fact that $F(\kappa)$ is holomorphic function on the strip $-2 < \Im\kappa < 2$. This completes the derivation of formula (1.6) defining the operator L_0 in the class of difference operators.

Let me show now that the operator L_0 defining through (1.6) acts onto generating function $f^{(\kappa)}$ in the way it requires by (2.2). Indeed,

$$\begin{aligned} L_0[f^{(\kappa)}(z)] &= \frac{1}{4}\left[\frac{\sqrt{\kappa(\kappa+2i)}}{(\kappa+2i)\sqrt{\frac{2}{\kappa+2i}\sinh(\frac{\pi\kappa}{2})e^{i\pi}}}\left(1-e^{-(\kappa+2i)\arctan z}\right)\right. \\ &\quad \left.+\frac{\sqrt{\kappa(\kappa-2i)}}{(\kappa-2i)\sqrt{\frac{2}{\kappa-2i}\sinh(\frac{\pi\kappa}{2})e^{-i\pi}}}\left(1-e^{-(\kappa-2i)\arctan z}\right)\right] \\ &= \frac{i}{4}e^{-\kappa\arctan z}\left(e^{-2i\arctan z}-e^{2i\arctan z}\right) = \frac{z}{1+z^2}\,e^{-\kappa\arctan z} = z\frac{d}{dz}\,f^{(\kappa)}(z). \end{aligned} \tag{2.12}$$

3 Operators L_1 and L_{-1}

The operators L_1 and L_{-1} are related by hermitian conjugation. Therefore it is enough to find the operator L_{-1}. As it is done in the previous section we restrict our considerations to the zero momentum sector. The action of L_{-1} on the generating function $f^{(\kappa)}(z)$ is given by the formula

$$L_{-1}[f^{(\kappa)}(z)] = -\frac{1}{\mathcal{N}(\kappa)} + \frac{d}{dz}\,f^{(\kappa)}(z). \tag{3.1}$$

Therefore formally its kernel can be written as

$$L_{-1}(\kappa,\kappa') \sim \sum_{m=1}^{\infty} v_{m+1}^{(\kappa)}\,\sqrt{(m+1)m}\,v_m^{(\kappa')} \tag{3.2}$$

As it is explained in the Introduction the operator L_{-1} is completely determined by its action onto the one-particle Fock space. Therefore in order to find it we will use the same method as the one used in the previous section. There will be only one modification of the method related to the fact that the operator L_1 (conjugated to the operator L_{-1}) has a one-dimensional kernel. First, we find an operator G_{-1} such that it "inverts" L_{-1} on the one-particle Fock space. More precisely G_{-1} satisfies the equation

$$G_{-1}[L_{-1}[f]](\kappa) = f(\kappa) \quad \text{and} \quad L_{-1}[G_{-1}[f]](\kappa) = P_1[f](\kappa), \tag{3.3a}$$

where P_1 is a projector on the subspace on which L_1 is a non-degenerate operator:

$$P_1[f](\kappa) = f(\kappa) - v_1^{(\kappa)}\int_{-\infty}^{\infty} d\kappa'\,v_1^{(\kappa')}f(\kappa'). \tag{3.3b}$$

From equation (3.2) one can easily obtain the formal expression for the kernel of operator G_{-1}:

$$G_{-1}(\kappa,\kappa') = \sum_{m=1}^{\infty} v_m^{(\kappa)} \frac{1}{\sqrt{m(m+1)}} v_{m+1}^{(\kappa')}. \tag{3.4}$$

Straightforward calculations (very similar to the ones presented in Appendix A) show that this expression does indeed define a distribution:

$$G_{-1}(\kappa,\kappa') = \frac{1}{\kappa\sqrt{\mathcal{N}(\kappa)\mathcal{N}(\kappa')}} - \frac{1}{2\kappa}\sqrt{\frac{\mathcal{N}(\kappa)}{\mathcal{N}(\kappa')}}\,\frac{\kappa-\kappa'}{\sinh\frac{\pi}{2}(\kappa-\kappa')} \tag{3.5}$$

and therefore G_{-1} is a kernel operator.

Second, we find the "inverse" of operator G_{-1} on the one particle Fock space. In other words we are going to solve the first equation in (3.3a). The resulting operator will coincide with the operator L_{-1} on the one-particle Fock space. Since L_{-1} is completely determined by its action on this subspace we will actually find operator L_{-1} on the whole Fock space.

So we need to solve the equation

$$\int_{-\infty}^{\infty} d\kappa' \,\frac{1}{\kappa\sqrt{\mathcal{N}(\kappa)\mathcal{N}(\kappa')}}\, g(\kappa') - \frac{1}{2\kappa}\int_{-\infty}^{\infty} d\kappa' \sqrt{\frac{\mathcal{N}(\kappa)}{\mathcal{N}(\kappa')}}\,\frac{\kappa-\kappa'}{\sinh\frac{\pi}{2}(\kappa-\kappa')}\, g(\kappa') = f(\kappa), \tag{3.6}$$

where $g(\kappa') = L_{-1}[f](\kappa')$. Notice now that by the construction g is a function that belongs to the image of L_{-1}, therefore it satisfies the equation $P_1[g] = g$. This equation just reflects the fact that there is no state $a_1^\dagger|0\rangle$ in the image of operator L_{-1} restricted to the one-particle subspace. Eventually the first term in the equation is identically zero[2]. It is useful to introduce new functions F and G

$$G(\kappa') = \sqrt{\mathcal{N}(\kappa')}\, g(\kappa') \quad \text{and} \quad F(\kappa) = \kappa\sqrt{\mathcal{N}(\kappa)}\, f(\kappa) \tag{3.7}$$

for which the equation takes extremely simple form

$$-\frac{1}{2}\int_{-\infty}^{\infty} d\kappa' \,\frac{\kappa'-\kappa}{\sinh\frac{\pi}{2}(\kappa'-\kappa)}\, G(\kappa') = F(\kappa).$$

This equation can be solved using method of Fourier transform. Using that

$$\int_{-\infty}^{\infty} d\kappa'\, e^{i\omega\kappa'} \frac{\kappa'}{\sinh\frac{\pi}{2}\kappa'} = \frac{2}{\cosh^2\omega}$$

one can easily obtain the solution

$$G(\kappa) = -\frac{1}{2}F(\kappa) - \frac{1}{4\pi}\int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\kappa'\, e^{i\omega(\kappa-\kappa')} \cosh 2\omega F(\kappa').$$

[2]We use the fact that $v_1^{(\kappa')} = \frac{1}{\sqrt{\mathcal{N}(\kappa')}}$

Using the arguments we presented equation (2.10) one obtains

$$G(\kappa) = -\frac{1}{2}\, F(\kappa) - \frac{1}{4}\Big[F(\kappa + i2^-) + F(\kappa - i2^-)\Big].$$

Here the function $F(\kappa)$ is supposed to be holomorphic on the strip $-2 < \Im\kappa < 2$. Now using the relations (3.7) one obtains expression (1.8) defining the operator L_{-1} in a class of difference operators. An expression for the operator L_1 is easily obtained by hermitian conjugation.

Using the same technique as we used in deriving (2.12) we can show that operators L_1 and L_{-1} defined via (1.8) act on the generating function (1.1) as follows

$$L_1[f^{(\kappa)}(z)] = z^2\frac{d}{dz}\, f^{(\kappa)}(z) \tag{3.8a}$$

$$L_{-1}[f^{(\kappa)}(z)] = -\frac{1}{\mathcal{N}(\kappa)} + \frac{d}{dz}\, f^{(\kappa)}(z) \tag{3.8b}$$

4 Commutation relations

In this section we are going to show that operators L_0, $L_{\pm 1}$ represented by the difference operators (1.6) and (1.8) satisfy the standard commutation relations.

First, let me demonstrate how to calculate the commutator of L_1 and L_{-1}. From (1.8) we obtain the identities

$$L_1[L_{-1}[g]](\kappa) = -\frac{\kappa}{2}\, L_{-1}[g] + \frac{i}{4}\left[\sqrt{\kappa(\kappa+2i)}L_{-1}[g](\kappa+2i) - \sqrt{\kappa(\kappa-2i)}\, L_{-1}[g](\kappa-2i)\right]; \tag{4.1a}$$

$$L_{-1}[L_1[g]](\kappa) = -\frac{\kappa}{2}\, L_1[g] - \frac{i}{4}\left[\sqrt{\kappa(\kappa+2i)}L_1[g](\kappa+2i) - \sqrt{\kappa(\kappa-2i)}\, L_1[g](\kappa-2i)\right]. \tag{4.1b}$$

By subtracting the last equation from the first one one obtains

$$\begin{aligned}[L_1,\, L_{-1}][g](\kappa) &= \frac{\kappa}{2}\big(L_1[g] - L_{-1}[g]\big)(\kappa)\\ &+ \frac{i}{4}\left[\sqrt{\kappa(\kappa+2i)}\big(L_1[g] + L_{-1}[g]\big)(\kappa+2i) - \sqrt{\kappa(\kappa-2i)}\big(L_1[g] + L_{-1}[g]\big)(\kappa-2i)\right]\end{aligned}$$

Substitution of (1.8) and simplification yield

$$[L_1, L_{-1}][g] = 2L_0[g]. \tag{4.2}$$

Second , we check commutation relations between L_0 and L_1.

$$\begin{aligned}[L_0,\, L_1][g](\kappa) &= \frac{\kappa}{2}L_0[g](\kappa)\\ &+ \frac{1}{4}\left[\sqrt{\kappa(\kappa+2i)}\big(L_1[g] - iL_0[g]\big)(\kappa+2i) + \sqrt{\kappa(\kappa-2i)}\big(L_1[g] + iL_0[g]\big)(\kappa-2i)\right]\end{aligned} \tag{4.3}$$

Substitution of (1.6) and (1.8) and simplification yield

$$[L_0,\, L_1][g] = -L_1[g]. \tag{4.4}$$

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Note added: While this paper was nearing completion, the paper [10] appeared, which contains in Section 5 similar expressions for L_0 and $L_{\pm 1}$. The reason for publishing this paper is mainly to give another point of view on the Virasoro operators in the κ-basis.

Appendix

A Calculations of G_0

The kernel for operator G_0 can be written as the following contour integral

$$G_0(\kappa,\kappa') = \frac{1}{\kappa\kappa'\sqrt{\mathcal{N}(\kappa)\mathcal{N}(\kappa')}}\frac{1}{2\pi i}\oint_C \frac{dz}{z}\,(1-e^{-\kappa\arctan\frac{1}{z}})(1-e^{-\kappa'\arctan z}). \tag{A.1}$$

Notice that this integral has two logarithmic singularities at the points $\pm i$. Therefore the complex plane has two cuts starting at those points. We choose them to lie on the imaginary axis (for more details on the contour see Appendix in [8]). To calculate the integral we deform the contour C in such way that it will go along the cuts. We introduce $z = ix-\epsilon$, $z = ix+\epsilon$, $z = -ix+\epsilon$ and $z = -ix-\epsilon$ on contours C_+, C_-, C'_+ and C'_- respectively

$$\begin{aligned}\kappa\kappa'\sqrt{\mathcal{N}(\kappa)\mathcal{N}(\kappa')}\,G_0(\kappa,\kappa') = \frac{1}{2\pi i}\int_1^\infty \Bigg[&\frac{idx}{ix-\epsilon}\left(1-e^{-\kappa\arctan\frac{1}{ix-\epsilon}}\right)\left(1-e^{-\kappa'\arctan(ix-\epsilon)}\right)\\ &+\frac{-idx}{ix+\epsilon}\left(1-e^{-\kappa\arctan\frac{1}{ix+\epsilon}}\right)\left(1-e^{-\kappa'\arctan(ix+\epsilon)}\right)\\ &+\frac{-idx}{-ix+\epsilon}\left(1-e^{-\kappa\arctan\frac{1}{-ix+\epsilon}}\right)\left(1-e^{-\kappa'\arctan(-ix+\epsilon)}\right)\\ &+\frac{idx}{-ix-\epsilon}\left(1-e^{-\kappa\arctan\frac{1}{-ix-\epsilon}}\right)\left(1-e^{-\kappa'\arctan(-ix-\epsilon)}\right)\Bigg]\end{aligned}$$

Using that $\operatorname{arctanh}(ix\pm\epsilon) = \pm\frac{\pi}{2}+i\coth^{-1}x$ and $\operatorname{arctanh}(ix\pm\epsilon)^{-1} = -i\coth^{-1}x$ one obtains

$$\begin{aligned}= \frac{1}{2\pi i}\int_1^\infty \frac{dx}{x}\Bigg[&\left(1-e^{-\kappa(-i\coth^{-1}x)}\right)\left(1-e^{-\kappa'(-\frac{\pi}{2}+i\coth^{-1}x)}\right)\\ &-\left(1-e^{-\kappa(-i\coth^{-1}x)}\right)\left(1-e^{-\kappa'(\frac{\pi}{2}+i\coth^{-1}x)}\right)\\ &+\left(1-e^{-\kappa i\coth^{-1}x}\right)\left(1-e^{-\kappa'(\frac{\pi}{2}-i\coth^{-1}x)}\right)\\ &-\left(1-e^{-\kappa i\coth^{-1}x}\right)\left(1-e^{-\kappa'(-\frac{\pi}{2}-i\coth^{-1}x)}\right)\Bigg]\end{aligned}$$

Now the change of variable

$$x = \coth u \quad \text{and} \quad dx = -\frac{du}{\sinh^2 u}$$

yields

$$= \frac{1}{\pi i} \int_0^\infty \frac{du}{\sinh 2u} \Big[\Big(1 - e^{i\kappa u}\Big) \Big(1 - e^{\frac{\pi\kappa'}{2} - i\kappa' u}\Big) \\ - \Big(1 - e^{i\kappa u}\Big) \Big(1 - e^{-\frac{\pi\kappa'}{2} - i\kappa' u}\Big) \\ + \Big(1 - e^{-i\kappa u}\Big) \Big(1 - e^{-\frac{\pi\kappa'}{2} + i\kappa' u}\Big) \\ - \Big(1 - e^{-i\kappa u}\Big) \Big(1 - e^{\frac{\pi\kappa'}{2} + i\kappa' u}\Big) \Big]$$

Simplification yields

$$= \frac{2}{\pi i} \sinh \frac{\pi\kappa'}{2} \int_0^\infty \frac{du}{\sinh 2u} \left[e^{i\kappa' u} - e^{i(\kappa'-\kappa)u} - e^{-i\kappa' u} + e^{-i(\kappa'-\kappa)u} \right] \\ = \frac{2}{\pi i} \sinh \frac{\pi\kappa'}{2} \int_{-\infty}^\infty du \, \mathcal{P} \frac{1}{\sinh 2u} \left[e^{i\kappa' u} + e^{-i(\kappa'-\kappa)u} \right] \quad (A.2)$$

Using the fact that

$$\frac{1}{\pi i} \int_{-\infty}^\infty du \, \mathcal{P} \frac{1}{\sinh 2u} \, e^{i\beta u} = \frac{1}{2} \tanh \frac{\pi\beta}{4} \quad (A.3)$$

one obtains

$$= \sinh \frac{\pi\kappa'}{2} \left[\tanh \frac{\pi\kappa'}{4} + \tanh \frac{\pi(\kappa-\kappa')}{4} \right] = 2 \, \frac{\sinh \frac{\pi\kappa}{4} \sinh \frac{\pi\kappa'}{4}}{\cosh \frac{\pi(\kappa-\kappa')}{4}}. \quad (A.4)$$

Finally one gets

$$G_0(\kappa, \kappa') = \left[\frac{\theta(\kappa)}{\kappa} \right]^{1/2} \left[\frac{\theta(\kappa')}{\kappa'} \right]^{1/2} \frac{1}{4 \cosh \left[\frac{\pi}{4}(\kappa - \kappa') \right]}. \quad (A.5)$$

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